

ON INTEGRAL OF EXPONENT OF A HOMOGENEOUS POLYNOMIAL

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ABSTRACT. We introduce the notion of G -hypergeometric function, where G is a complex Lie group. In the case when G is a complex torus, this notion amounts to the notion of Gelfand's A -hypergeometric function. We show that the integral $\int e^{P(x_1, \dots, x_n)} dx_1 \dots dx_n$, where P is a homogeneous polynomial, is a $GL(n)$ -hypergeometric function of algebraic $SL(n)$ -invariants of the polynomial.

1. Introduction. The problem of computing the integral of exponent of a homogeneous form of n variables over n -dimensional space has been posed by V. V. Dolotin [1]. He also developed the new techniques of invariant theory (diagram techniques, discriminant theory), which allows one to compute invariants explicitly [2–4]. The paper [5] by I. M. Gelfand and M. I. Graev implies that the integral of exponent of a homogeneous form is a generalized hypergeometric function of the coefficients of the form. In the paper [6] by A. Yu. Morozov and Sh. R. Shakirov this integral is calculated explicitly in several particular cases as a function of algebraic $SL(n)$ -invariants. In these particular cases the integral is a generalized hypergeometric function, in the sense of I. M. Gelfand, of basic invariants. The authors of [6] conjecture that the integral is always a generalized hypergeometric function of algebraic invariants of a form. The purpose of the present paper is to discuss this conjecture. Recall that the notion of generalized hypergeometric function in the sense of Gelfand is related with an action of complex torus. We generalize this notion to non-Abelian complex Lie groups. We call the obtained functions by G -hypergeometric functions, where G is a Lie group. In the case when G is a torus, this notion essentially coincides with the notion of Gelfand's A -hypergeometric function. We show that integral of exponent of a form as a function of basic invariants is always a $GL(n)$ -hypergeometric function. Thus, in order to make the Morozov–Shakirov conjecture true, one should extend the notion of A -hypergeometric function to G -hypergeometric function. There are very interesting open problems of explicit computation of G -hypergeometric functions, their equations and their singularities, in particular, for

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integral of exponent of a form. Interesting computations can be found in [12,13]. Other generalizations of hypergeometric functions to Lie groups are contained in [5,14].

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2. Preliminaries.

Notations. $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, $k_i \geq 0$, $x^k = x_1^{k_1} \dots x_n^{k_n}$.

A is the set of all k of one and the same homogeneous degree $d = \sum_{i=1}^n k_i$.

$P(a; x) = \sum_{k \in A} a_k x^k$ is a homogeneous polynomial, where $a = (a_k)_{k \in A} \in \mathbb{C}^A$.

$J_{n|d}(C; a) = \int_C e^{P(a; x)} dx$ is the integral over an n -dimensional real contour $C \subset \mathbb{C}^n$, on which the function $e^{P(a; x)}$ rapidly decreases at infinity.

$\Delta(A)$ is the discriminant, i. e. the set of $a \in \mathbb{C}^A$ for which the projective hypersurface $P(a; x) = 0$ is singular.

For each face Γ of the polytop $\text{Conv}(A)$, the convex hull of the set A (including the polytop itself), denote by $\pi_\Gamma : \mathbb{C}^A \rightarrow \mathbb{C}^{\Gamma \cap A}$ the projection, $\pi_\Gamma(a) = (a_k)_{k \in \Gamma \cap A}$, and let $E(\Gamma) = \pi_\Gamma^{-1}(\Delta(\Gamma \cap A))$. Let E be the union of $E(\Gamma)$ for all Γ . E is called the *principal A -discriminant* [9].

Theorem 1 (corollary of [5]). *The integral $J_{n|d}(C; a)$ satisfies the A -hypergeometric system of equations [7].*

Theorem 2. *The singularities of the function $J_{n|d}(C; a)$ lie on the discriminant $\Delta(A)$.*

Proof. From Theorem 1, from computation of the characteristic variety of the A -hypergeometric system [7,8], and from the description of the varieties projectively dual to the closures of the torus orbits [9, Introduction], it follows that the singularities of the function $J_{n|d}(C; a)$ lie on the principal A -discriminant E . Since for $\Gamma \neq \text{Conv}(A)$ the variety $E(\Gamma)$ is not $SL(n)$ -invariant and the function $J_{n|d}(C; a)$ is $SL(n)$ -invariant, this implies that actually the singularities lie only on the variety $E(\text{Conv}(A)) = \Delta(A)$. Q. E. D.

3. The G -hypergeometric D -module.

In this section we follow the paper [7, §2], generalizing some of its results to the case of non-Abelian Lie group G .

Let V be a finite dimensional representation of the group G . Assume that the unit operator on V lies in the image of the Lie algebra \mathfrak{g} of the group G (otherwise let us replace G by $G \times \mathbb{C}^*$). Let $W = \overline{G \cdot v}$ be the closure of the G -orbit of a nonzero vector $v \in V$. Assume that

this closure contains finitely many G -orbits. Let V^* be the dual G -module. Let $\beta : \mathfrak{g} \rightarrow \mathbb{C}$ be a character of the Lie algebra. Consider the D_V -module $M = D_V / (D_V \cdot J_W + D_V \cdot (L_{e_i} - \beta(e_i))_{i=1}^{\dim \mathfrak{g}})$, where J_W is the ideal of the variety W , e_i is a basis of the Lie algebra \mathfrak{g} , L_{e_i} is the Lie derivative. This D -module has been considered in the book [10, Theorem 5.2.12] for W smooth, where it is proved that the characteristic variety $SS(M)$ is contained in the union of conormal bundles $T_{W_\alpha}^* V$ to the G -orbits W_α in W . For W non-smooth the proof is similar ([7]). Hence the D -module M is holonomic.

Consider the Fourier transform FM of the D_V -module M . This D_{V^*} -module corresponds to the system of equations

$$(1) \quad \square_j \varphi = 0,$$

$$(2) \quad (L_{e_i} + \beta(e_i) - \chi_0(e_i)) \varphi = 0$$

for a function $\varphi(w)$ on the space V^* . Here \square_j is the differential operator with constant coefficients corresponding to a polynomial equation of the variety W under the Fourier transform; $\chi_0(e_i)$ is the trace of the action of e_i on V . Since the D_V -module M is monodromic [11] and G_1 -equivariant, where $G_1 \subset G$ is the Lie group with the Lie algebra $\mathfrak{g}_1 = \text{Ker } \beta$, the same is true also for the D_{V^*} -module FM . (In particular, G_1 contains the commutator $[G, G]$.) Therefore, the characteristic variety $SS(FM)$ corresponds to $SS(M)$ under the identification $T^*V^* \simeq V \times V^* \simeq T^*V$.

Definition. *The system of equations (1,2) on the space V^* and the corresponding system on the variety of orbits V^*/G_1 is called the G -hypergeometric system, and its holomorphic solutions on an open subset in V^* or in V^*/G_1 are called G -hypergeometric functions.*

Theorem 3. *The G -hypergeometric system has a finite number of solutions at the general point. The singularities of the solutions lie on the union of conic varieties whose projectivization is projectively dual to the projectivization of the varieties W_α , and has codimension 1.*

Proof. This follows from the holonomicity of the D -module FM and from the description of $SS(FM)$ given above (as in [7]).

Theorem 4. *The integral $\varphi(a) = J_{n|d}(C; a)$ is a $GL(n)$ -hypergeometric function on the space $V^* = \mathbb{C}^A$.*

Proof. Equation (1) follows from the fact that each polynomial equation of the variety $W = \overline{G \cdot v} \subset V$, where $V = (\mathbb{C}^A)^*$, $G = GL(n)$, $v = (1, 1, \dots, 1)$, vanishes also on the variety $\overline{(\mathbb{C}^*)^n \cdot v}$, which coincides with W in this case. Hence system of equations (1) is equivalent to the

first part of the A -hypergeometric system. Equation (2) follows from integration by parts. Q. E. D.

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